

# EQUICHARACTERISTIC ÉTALE COHOMOLOGY IN DIMENSION ONE

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**ABSTRACT.** The Grothendieck-Ogg-Shafarevich formula expresses the Euler characteristic of an étale sheaf on a curve in terms of local data. The purpose of this paper is to prove a version of the G-O-S formula which applies to equicharacteristic sheaves (a bound, rather than an equality). This follows a proposal of R. Pink.

The basis for the result is the characteristic- $p$  “Riemann-Hilbert” correspondence, which relates equicharacteristic étale sheaves to  $\mathcal{O}_{F,X}$ -modules. In the paper we prove a version of this correspondence for curves, considering both local and global settings. In the process we define an invariant, the “minimal root index,” which measures the local complexity of an  $\mathcal{O}_{F,X}$ -module. This invariant provides the local terms for the main result.

## 1. INTRODUCTION

**1.1. Overview of paper.** We are concerned with computing sizes of étale cohomology groups in positive characteristic. Fix a base field,  $k$ , which is algebraically closed and has characteristic  $p > 0$ .

In general, the properties of étale cohomology groups over  $k$  depend heavily on what coefficient ring one chooses to use. One can assume that the coefficient ring is  $p$ -torsion (or  $p^r$ -torsion), or one can assume that the coefficient ring has a characteristic which is coprime to  $p$ . It is typical to separate these two cases. The first case (the “equicharacteristic” case) seems to be less tractable than the second. But a number of motivations for studying the second case do carry over to the first. For example, it is known that zeta functions modulo  $p$  can be computed from equicharacteristic étale cohomology groups. (See [6].)

A good tool for computing sizes of étale cohomology groups is the “Grothendieck-Ogg-Shafarevich formula.” Let  $Y$  be a smooth projective  $k$ -curve, and let  $N$  be a constructible étale sheaf of  $\mathbb{F}_\ell$ -modules on  $Y$ , where  $\ell$  denotes a prime different from  $p$ . Assume that the sheaf  $N$  is locally constant on an open subset  $X \subseteq Y$ , and that its stalks are zero at points outside of  $X$ . Then,

$$(1.1.1) \quad \chi(Y, N) = (2 - 2g)n - \sum_{y \in |Y \setminus X|} (n + Sw_y(N)).$$

In this formula,  $\chi(Y, N)$  denotes the Euler characteristic of  $N$  (the alternating sum of the dimensions  $h^i(Y, N)$ ),  $g$  denotes the genus of  $Y$ , and  $n$  denotes the generic rank of  $N$ . The expression  $Sw_y(N)$  denotes the “Swan conductor,” an invariant

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which measures the local ramification of the sheaf  $N$ . (See [10] for a discussion of this formula, including an application to surfaces.)

It is natural to ask whether a similar formula could be constructed for the equicharacteristic case. This question leads to some difficulties, however, which are tied up with the unpredictable behavior of  $H^1(Y, N)$ . It is possible to construct an example of two  $\mathbb{F}_p$ -sheaves  $N$  and  $N'$  on the same curve  $Y$ , both having the same rank and local ramification, but which nonetheless have different Euler characteristics. (See Example 4.5.10 in this paper.) So clearly, an exact formula for  $\chi(Y, N)$  based on local information about  $N$  will not be possible.<sup>1</sup>

A good compromise is to construct a *lower bound* for  $\chi(Y, N)$  in the equicharacteristic case. This idea was proposed by R. Pink. Pink himself proved a lower bound for  $\chi(Y, N)$  which applies under some restrictions on the wild ramification of  $N$  (Theorem 0.2 in [9]). The purpose of this paper is to prove the following general extension of Pink's theorem.

**Theorem 1.1.2.** *Let  $Y \rightarrow \text{Spec } k$  be a smooth projective  $k$ -curve of genus  $g$ . Let  $M$  be a rank- $n$  constructible étale sheaf of  $\mathbb{F}_p$ -modules on  $Y$  in which all sections have open support. Then,*

$$(1.1.3) \quad \chi(Y, M) \geq (1 - g)n - \sum_{y \in Y(k)} \mathfrak{C}(M_{(y)}).$$

In the expression above,  $\mathfrak{C}(M_{(y)})$  denotes a local invariant which we call the “minimal root index.”

The proof of Theorem 1.1.2 is based on a study of the relationship between étale  $\mathbb{F}_p$ -sheaves and quasi-coherent  $\mathcal{O}_Y$ -modules. Suppose that  $M$  is the sheaf defined in the theorem. Let

$$(1.1.4) \quad \mathcal{M} = \mathcal{H}om_{\mathbb{F}_p}(M, \mathcal{O}_{Y_{\text{ét}}}).$$

Then  $\mathcal{M}$  has the structure of an  $\mathcal{O}_Y$ -module. Additionally, the  $p$ th-power map on  $\mathcal{O}_{Y_{\text{ét}}}$  determines a Frobenius-linear endomorphism  $F: \mathcal{M} \rightarrow \mathcal{M}$ . From the data of  $\mathcal{M}$  together with this endomorphism, one can recover the original sheaf  $M$ . This association is part of the characteristic- $p$  “Riemann-Hilbert” correspondence of M. Emerton and M. Kisin ([3]). Following their terminology, we call  $\mathcal{M}$  an “ $\mathcal{O}_{F,Y}$ -module.”

The characteristic- $p$  Riemann-Hilbert correspondence is developed in full generality in [3]. Since we prefer to avoid the language of derived categories, we will not make direct use of the results from that paper. We construct a miniature version of the correspondence which applies to  $\mathbb{F}_p$ -sheaves on  $k$ -curves. Our version includes a localization functor. The key results are Theorem 3.3.7 and Propositions 4.1.1, 4.2.2, 4.2.14, and 4.2.16.

A key ingredient for the construction of Theorem 1.1.2—the ingredient, in fact, that allows the extension beyond Pink's original result—is the notion of a “root” of an  $\mathcal{O}_{F,Y}$ -module. This notion is due to G. Lyubeznik (see [8]). A “root” for  $\mathcal{M}$  is a special type of coherent generating submodule. If  $\mathcal{M}_0 \subseteq \mathcal{M}$  is a root, then the images of  $\mathcal{M}_0$  under repeated applications of the map  $F$  determine an ascending filtration for the sheaf  $\mathcal{M}$ . (See Definition 2.2.1 and Proposition 2.2.3). In this

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<sup>1</sup>An exact formula may be possible if we allow global data for the sheaf  $N$ . The work of W. A. Hawkins in [4] and [5] contains results in that direction.

paper we develop the properties of roots in parallel with the construction of the Riemann-Hilbert correspondence.

One interesting result that arises is Theorem 4.3.1, which asserts the existence of canonical minimal roots for  $\mathcal{O}_{F,X}$ -modules in dimension one. (This result is critical for Theorem 1.1.2. The local term  $\mathfrak{C}(M_{(y)})$  is based on a measurement of the minimal root of  $\mathcal{M}$ .) I am pleased to point out that a much stronger version of this result has been proven, independently, by M. Blickle ([2]). Blickle proved the existence of canonical minimal roots over any  $F$ -finite regular ring. In particular, this means that minimal roots exist on smooth  $k$ -schemes of arbitrary dimension. (This naturally suggests an extension of the definition of  $\mathfrak{C}!$ )

Section 2 in this paper reviews terminology for  $\mathcal{O}_{F,X}$ -modules and establishes some basic results. Sections 3 and 4 develop the one-dimensional Riemann-Hilbert correspondence. (Section 3 contains local results, and Section 4 contains global results.) Theorem 1.1.2 appears in subsection 4.4. (The proof is similar to the one used in [9].) The paper closes with three examples involving sheaves on the projective line.

**1.2. Further directions.** A natural goal is to determine conditions under which formula (1.1.3) yields an equality. Empirical evidence suggests that equality occurs “generically,” and it would be interesting to make that statement more precise. Another goal is to gain a better understanding of the local invariant  $\mathfrak{C}(M_{(y)})$ . The invariant is defined in this paper in terms  $\mathcal{O}_{F,Y}$ -modules, but it should be possible to understand it directly in terms of the localization  $M_{(y)}$  (which is a sheaf on  $\mathrm{Spec} \mathcal{O}_{Y_{\text{ét}},y}$ ). The localization  $M_{(y)}$  is essentially an equicharacteristic Galois representation.

Another direction has to do with  $p$ -adic cohomology. Theorem 1.1.2 is an assertion about  $p$ -torsion sheaves, but without much difficulty it can be converted into a statement about étale  $\mathbb{Q}_p$ -sheaves. The  $\mathbb{Q}_p$ -version of the theorem might have interesting connections with other known results on  $p$ -adic cohomology. (Consider for example Theorem 4.3.1 in [7], which is a Grothendieck-Ogg-Shafarevich formula for rigid cohomology.)

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**1.4. Notation and conventions.** Throughout this paper,  $p$  denotes a prime,  $r$  denotes a positive integer, and  $k$  denotes an algebraically closed field of characteristic  $p$ . All sheaves are assumed to be sheaves on an étale site. Thus, if  $X$  is a  $k$ -scheme,  $\mathcal{O}_X$  denotes the étale coordinate sheaf on  $X$ . If  $x$  is a  $k$ -point of  $X$ , then  $\mathcal{O}_{X,x}$  denotes the (étale) stalk of  $\mathcal{O}_X$  at  $x$ .

If  $R$  is a ring, let  $\mathbf{LMod}(R)$  (or simply  $\mathbf{Mod}(R)$ , if  $R$  is commutative) denote the category of left  $R$ -modules. If  $X$  is a scheme and  $\mathcal{R}$  is a sheaf of rings on  $X$ , let  $(\mathbf{L})\mathbf{Mod}(X, \mathcal{R})$  denote the category of (left)  $\mathcal{R}$ -modules on  $X$ .

If  $S$  is a  $k$ -algebra, let  $F_S: S \rightarrow S$  denote the Frobenius map. If  $Z$  is a  $k$ -scheme, let  $F_Z: Z \rightarrow Z$  denote the Frobenius morphism.

All schemes are assumed to be noetherian and separated.

## 2. $\mathcal{O}_{F^r, X}$ -MODULES

Let  $X$  be a  $k$ -scheme. This section is concerned with quasi-coherent  $\mathcal{O}_X$ -modules that have Frobenius-linear endomorphisms. For the study of these modules it is convenient to introduce the sheaf  $\mathcal{O}_{F^r, X}$ . This is a sheaf of noncommutative rings in which the multiplication rule is determined by the  $r$ th Frobenius map on  $\mathcal{O}_X$ . Notation and terminology in this section are borrowed from [3].

**2.1. The category of left  $\mathcal{O}_{F^r, X}$ -modules.** Let  $S$  be a  $k$ -algebra. Then  $S[F^r]$  is the twisted polynomial ring determined by the  $r$ th Frobenius endomorphism,

$$(2.1.1) \quad (F_S)^r: S \rightarrow S.$$

Elements of the ring  $S[F^r]$  may be expressed as finite sums of the form

$$(2.1.2) \quad \sum_{i \geq 0} s_i F^{ri},$$

and multiplication is determined by the rule  $F^r s = s^{(p^r)} F^r$ . Similarly, let  $X$  be a  $k$ -scheme. Then  $\mathcal{O}_{F^r, X}$  denotes the sheaf of twisted polynomial rings determined by  $F_X^r: X \rightarrow X$ . If  $\text{Spec } T \subseteq X$  is any affine open inside of  $X$ , then the sections of  $\mathcal{O}_{F^r, X}$  over  $\text{Spec } T$  form the ring  $T[F^r]$ .

This notation provides a convenient way to express Frobenius-linear endomorphisms. Let  $M$  be an  $S$ -module. Then a left  $S[F^r]$ -module structure of  $M$  is uniquely specified by an additive endomorphism  $\phi: M \rightarrow M$  satisfying  $\phi(sm) = s^{(p^r)} \phi(m)$  for all  $s \in S$ ,  $m \in M$ . (The map  $\phi$  expresses the left action of  $F^r$  on  $M$ .) Thus a left  $S[F^r]$ -module is simply an  $S$ -module with a Frobenius-linear endomorphism.

If  $\mathcal{M}$  is a left  $\mathcal{O}_{F^r, X}$ -module, then the sheaf endomorphism  $\mathcal{M} \rightarrow \mathcal{M}$  determined by the action of  $F^r$  determines a morphism

$$(2.1.3) \quad F_X^{r*} \mathcal{M} \rightarrow \mathcal{M}$$

which is  $\mathcal{O}_X$ -linear. We refer to this homomorphism as the structural morphism of  $\mathcal{M}$ .

**Definition 2.1.4.** A unit  $\mathcal{O}_{F^r, X}$ -module is a left  $\mathcal{O}_{F^r, X}$ -module which is quasi-coherent (as an  $\mathcal{O}_X$ -module) and whose structural morphism is an isomorphism.

The term “unit” may be similarly applied to modules over rings. A left  $S[F^r]$ -module is unit if its structural homomorphism  $F_S^{r*} M \rightarrow M$  is an isomorphism. Let  $\mathbf{LMod}^u(S[F^r])$  be the full subcategory of unit  $S[F^r]$ -modules in  $\mathbf{LMod}(S[F^r])$ . Additionally, let  $\mathbf{LMod}^{fu}(S[F^r])$  be the subcategory of  $\mathbf{LMod}^u(S[F^r])$  consisting of those objects which are finitely-generated left  $S[F^r]$ -modules. (We shall call these simply “finitely-generated unit  $S[F^r]$ -modules.”)

The categories  $\mathbf{LMod}^u(X, \mathcal{O}_{F^r, X})$  and  $\mathbf{LMod}^{fu}(X, \mathcal{O}_{F^r, X})$  are defined similarly. The category  $\mathbf{LMod}^u(X, \mathcal{O}_{F^r, X})$  consists of the unit  $\mathcal{O}_{F^r, X}$ -modules, and the category  $\mathbf{LMod}^{fu}(X, \mathcal{O}_{F^r, X})$  consists of those unit  $\mathcal{O}_{F^r, X}$ -modules which are finitely-generated once restricted to any affine open subset of  $X$ . Using terminology from [3], we refer to objects from  $\mathbf{LMod}^{fu}(X, \mathcal{O}_{F^r, X})$  as *locally finitely-generated*

unit (lfgu)  $\mathcal{O}_{F^r, X}$ -modules. (Note that “finitely-generated” refers to the left  $\mathcal{O}_{F^r, X}$ -module structure of the sheaf in question, not to its  $\mathcal{O}_X$ -module structure.)

Let  $f: X \rightarrow Y$  be a morphism of schemes over  $k$ , and let  $\mathcal{M}$  be a unit  $\mathcal{O}_{F^r, Y}$ -module. The  $\mathcal{O}_X$ -module pullback  $f^*\mathcal{M}$  of  $\mathcal{M}$  is given by

$$(2.1.5) \quad f^*\mathcal{M} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}.$$

There is a natural left  $\mathcal{O}_{F^r, X}$ -module structure on  $f^*\mathcal{M}$  which is expressed by the rule

$$(2.1.6) \quad F^r(f \otimes m) = f^{(p^r)} \otimes F^r(m).$$

**Proposition 2.1.7.** *Let  $f: X \rightarrow Y$  be a morphism of schemes over  $k$ , and let  $\mathcal{M}$  be a unit  $\mathcal{O}_{F^r, Y}$ -module. Then the pullback  $f^*\mathcal{M}$  is a unit  $\mathcal{O}_{F^r, X}$ -module. If  $\mathcal{M}$  is locally finitely-generated, then  $f^*\mathcal{M}$  is locally finitely-generated.*

*Proof.* The structural morphism of  $f^*\mathcal{M}$  is the composite of two morphisms,

$$(2.1.8) \quad F_X^{r*} f^* \mathcal{M} \rightarrow f^* F_Y^{r*} \mathcal{M} \rightarrow f^* \mathcal{M},$$

where the first arises from the commutative diagram

$$(2.1.9) \quad \begin{array}{ccc} X & \xrightarrow{F_X^r} & X \\ \downarrow f & & \downarrow f \\ Y & \xrightarrow{F_Y^r} & Y \end{array}$$

and the second is determined by the structural morphism for  $\mathcal{M}$ . Both maps are isomorphisms. Therefore  $f^*\mathcal{M}$  is a unit  $\mathcal{O}_{F^r, X}$ -module.

Now suppose that  $\mathcal{M}$  is locally finitely-generated. The condition that  $f^*\mathcal{M}$  is lfgu needs only to be checked locally. Choose any closed point  $x$  in  $X$ . Let  $V \subset Y$  be an affine open subscheme which contains  $f(x)$ . Let  $\{m_1, \dots, m_r\} \subseteq \mathcal{M}(V)$  be a set which generates  $\mathcal{M}|_V$  as a left  $\mathcal{O}_{F^r, V}$ -module. The pullbacks of  $\{m_1, \dots, m_r\}$  to  $f^{-1}(V)$  generate  $(f^*\mathcal{M})|_{f^{-1}(V)}$  as a left  $\mathcal{O}_{F^r, f^{-1}(V)}$ -module. Thus  $f^*\mathcal{M}$  is finitely-generated on an open neighborhood of  $x$ .  $\square$

**2.2. Roots of lfgu  $\mathcal{O}_{F^r, X}$ -modules.** While sheaves in the category  $\mathbf{LMod}^{f^u}(X, \mathcal{O}_{F^r, X})$  are not necessarily coherent, they have special coherent subsheaves which capture their structure. The concept of a root is due to Lyubeznik (see [8]).

**Definition 2.2.1.** *Let  $X$  be a  $k$ -scheme, and let  $\mathcal{M}$  be a unit  $\mathcal{O}_{F^r, X}$ -module. An  $\mathcal{O}_X$ -submodule  $\mathcal{M}' \subseteq \mathcal{M}$  is a root if*

- (1) *the  $\mathcal{O}_X$ -module  $\mathcal{M}'$  is coherent,*
- (2) *the  $\mathcal{O}_X$ -submodule of  $\mathcal{M}$  generated by  $F^r(\mathcal{M}')$  contains  $\mathcal{M}'$ , and*
- (3) *as a left  $\mathcal{O}_{F^r, X}$ -module,  $\mathcal{M}$  is generated by  $\mathcal{M}'$ .*

If a unit  $\mathcal{O}_{F^r, X}$ -module has a root, then it is locally finitely-generated. (This is easily deduced from properties 1 and 3 above.) The next proposition asserts that the converse is also true.

**Proposition 2.2.2.** *Let  $Y$  be a smooth  $k$ -scheme, and let  $\mathcal{M}$  be a unit  $\mathcal{O}_{F^r, Y}$ -module. If  $\mathcal{M}$  is locally finitely-generated, then it has a root.*

*Proof.* This is Theorem 6.1.3 from [3].  $\square$

The next two propositions will be useful in later parts of this paper. The first asserts that a root of a unit  $\mathcal{O}_{F^r, X}$ -module determines a filtration of the module by coherent subsheaves. The second proposition asserts that this filtration collapses on an open dense subset of the scheme  $X$ .

**Proposition 2.2.3.** *Let  $Y$  be a smooth  $k$ -scheme. Let  $\mathcal{M}$  be an lfgu  $\mathcal{O}_{F^r, Y}$ -module, and let  $\mathcal{M}_0 \subseteq \mathcal{M}$  be a root for  $\mathcal{M}$ . For each  $n \geq 1$ , let  $\mathcal{M}_n$  be the  $\mathcal{O}_X$ -submodule of  $\mathcal{M}$  generated by  $F^{rn}(\mathcal{M}_0)$ . The sequence  $(\mathcal{M}_n)$  is an ascending filtration of  $\mathcal{M}$ . For each  $n \geq 0$ , the structural morphism of  $\mathcal{M}$  determines a map*

$$(2.2.4) \quad F_X^{rn*} \mathcal{M}_0 \rightarrow \mathcal{M}_n$$

*which is an isomorphism.*

*Proof.* We prove the last assertion first. Since  $X$  is smooth, the Frobenius morphism  $F_X^{rn}: X \rightarrow X$  is flat and finite. Therefore the inclusion

$$(2.2.5) \quad \mathcal{M}_0 \hookrightarrow \mathcal{M}$$

induces an injection

$$(2.2.6) \quad F_X^{rn*} \mathcal{M}_0 \hookrightarrow F_X^{rn*} \mathcal{M}$$

Composing this injection with the  $n$ th power of the structural morphism of  $\mathcal{M}$  yields an injection

$$(2.2.7) \quad F_X^{rn*} \mathcal{M}_0 \hookrightarrow \mathcal{M}$$

whose image (by definition) is  $\mathcal{M}_n$ . Thus we obtain the desired isomorphism.

The assertion that  $(\mathcal{M}_n)$  is an ascending filtration follows from the observation that  $\mathcal{M}_{n+1} \subseteq \mathcal{M}$  is the sub- $\mathcal{O}_X$ -module generated by  $F^{rn}(\mathcal{M}_1)$ . Since  $\mathcal{M}_1$  contains  $\mathcal{M}_0$  (by property 2 of Definition 2.2.1),  $\mathcal{M}_{n+1}$  contains  $\mathcal{M}_n$ . Property 3 of Definition 2.2.1 implies that the union of the submodules  $\mathcal{M}_n$  is  $\mathcal{M}$ .  $\square$

**Proposition 2.2.8.** *Let  $Y$  be a smooth  $k$ -scheme. Let  $\mathcal{M}$  be an lfgu  $\mathcal{O}_{F^r, Y}$ -module, and let  $\mathcal{M}_0 \subseteq \mathcal{M}$  be a root for  $\mathcal{M}$ . Then there exists an open dense subset  $U \subseteq Y$  such that  $\mathcal{M}_{0|U} = \mathcal{M}_{|U}$ .*

*Proof.* Clearly we may assume that  $Y$  is irreducible (and therefore integral). Let  $(\mathcal{M}_n)$  be the filtration from Proposition 2.2.3. The isomorphism

$$(2.2.9) \quad F_Y^{r*} \mathcal{M}_0 \rightarrow \mathcal{M}_1$$

implies that the generic rank of  $\mathcal{M}_1$  is the same as the generic rank of  $\mathcal{M}_0$ . Therefore  $\mathcal{M}_1/\mathcal{M}_0$  is supported at a proper closed subset of  $Y$ . Let  $U \subseteq Y$  be the complement of this closed subset. The sheaf  $\mathcal{M}_{0|U}$  is stabilized by  $F^r$ . Since  $\mathcal{M}_{0|U}$  generates  $\mathcal{M}_{|U}$ ,  $\mathcal{M}_{0|U}$  must coincide with  $\mathcal{M}_{|U}$ .  $\square$

Lastly, we note that the reasoning used in the last two proofs also proves an important fact: any finitely-generated unit  $\mathcal{O}_{F^r, X}$ -module over a field must be finite-dimensional.

**Proposition 2.2.10.** *Let  $L$  be a field which contains  $k$ . Then a unit  $L[F^r]$ -module is finitely-generated if and only if it is finite-dimensional over  $L$ .*

*Proof.* Let  $V$  be a finitely-generated unit  $L[F^r]$ -module. Choose an algebraic closure  $\overline{L}$  for  $L$ . The pullback  $\overline{L} \otimes_L V$  is a finitely-generated unit  $\overline{L}[F^r]$ -module, which must have a root (by Theorem 6.1.3 from [3]). Let  $\overline{V}_0 \subseteq \overline{L} \otimes_L V$  be a root. As in the proof of Proposition 2.2.3, this root determines a filtration of  $\overline{L} \otimes V$ ,

$$(2.2.11) \quad \overline{V}_0 \subseteq \overline{V}_1 \subseteq \overline{V}_2 \subseteq \dots,$$

in which adjacent terms have isomorphisms  $F_{\overline{L}}^* \overline{V}_n \cong \overline{V}_{n+1}$ . Each term in this filtration must have the same (finite) dimension. Therefore  $\overline{L} \otimes V$  is finite-dimensional over  $\overline{L}$ , and  $V$  is finite-dimensional over  $L$ .

The converse is immediate.  $\square$

### 3. A LOCAL ANALYSIS OF $\mathcal{O}_{F^r, X}$ -MODULES IN DIMENSION ONE

Throughout this section, let  $A$  be a Henselization of the local ring  $k[t]_{(t)}$ . (For example, one can let  $A$  be the ring of elements of  $k[[t]]$  that are algebraic over  $k(t)$ .) Additionally, let  $K$  denote the fraction field of  $A$ . Note that if  $X$  is any smooth  $k$ -curve, then the stalk of its étale coordinate sheaf at any closed point is isomorphic to  $A$ . Thus unit  $\mathcal{O}_{F^r, X}$ -modules localize to unit  $A[F^r]$ -modules.

This section is concerned with finitely-generated unit  $A[F^r]$ -modules. We are primarily interested in those which are torsion-free as  $A$ -modules. The goals of this section are (1) to establish the local Riemann-Hilbert correspondence (Theorem 3.3.7), and (2) to define the “minimal root index” of a unit  $A[F^r]$ -module.

The following algebraic result is a starting point.

**Proposition 3.0.12.** *Let  $L$  be a separably closed field of characteristic  $p$ . Let  $H$  be a unit  $L[F^r]$ -module which is finite-dimensional over  $L$ . Then the set  $H^{(F^r)}$  of  $F^r$ -invariant elements of  $H$  spans  $H$ . This set forms an  $\mathbb{F}_{p^r}$ -vector space. The map*

$$(3.0.13) \quad H^{(F^r)} \otimes_{\mathbb{F}_{p^r}} L \rightarrow H$$

*is an isomorphism.*

*Proof.* This is a reformulation of Proposition 1.1 from [6].  $\square$

Note that this proposition implies that every unit  $L[F^r]$ -module which is finite-dimensional over  $L$  has an  $F^r$ -invariant basis.

**3.1. Trivializations of unit  $A[F^r]$ -modules.** Proposition 3.0.12 implies trivializations for unit  $A[F^r]$ -modules under certain assumptions. We state several assertions here for later use.

**Proposition 3.1.1.** *Let  $V$  be a finitely-generated unit  $K[F^r]$ -module, and let  $n$  be the dimension of  $V$  as a  $K$ -vector space. Let  $K^{sep}$  be a separable closure of the field  $K$ . Then there exists an isomorphism of left  $K^{sep}[F^r]$  modules,*

$$(3.1.2) \quad K^{sep} \otimes_K V \cong (K^{sep})^{\oplus n}.$$

*(The left  $K^{sep}[F^r]$ -module structure for the vector space on the right is given by the Frobenius map  $F_{K^{sep}}^r$ .)*

*Proof.* The module  $K^{sep} \otimes_K V$  is a unit  $K^{sep}[F^r]$ -module which is finite-dimensional over  $K^{sep}$ . By Proposition 3.0.12 it has  $K^{sep}$ -basis which is  $F^r$ -invariant. This basis determines the isomorphism.  $\square$

**Corollary 3.1.3.** *For some finite separable field extension  $K'/K$ , there exists an isomorphism of left  $K'[F^r]$ -modules*

$$(3.1.4) \quad K' \otimes_K V \cong (K')^{\oplus n}.$$

*Proof.* Let  $B \subseteq K^{sep} \otimes_K V$  be the basis which determines isomorphism (3.1.2). Simply choose  $K' \subseteq K^{sep}$  large enough that  $K' \otimes_K V \subseteq K^{sep} \otimes_K V$  contains  $B$ .  $\square$

**Proposition 3.1.5.** *Let  $W$  be a torsion-free unit  $A[F^r]$ -module such  $K \otimes_A W$  is isomorphic to  $K^{\oplus n}$  as a left  $A[F^r]$ -module. Then there exists an isomorphism*

$$(3.1.6) \quad W \cong K^{\oplus m} \oplus A^{\oplus n-m}$$

with  $0 \leq m \leq n$ .

*Proof.* Let us consider  $W$  as a submodule of  $K \otimes W$ . Let  $\{w_1, \dots, w_n\} \subseteq K \otimes W$  be the basis which determines the isomorphism to  $K^{\oplus n}$ . This basis is  $F^r$ -invariant. Note that  $t^n w_1 \in W$  for sufficiently large  $n$ . I claim that in fact  $n = 0$  is sufficient. For, suppose not: then  $t^N w_1 \in W$  and  $t^{N-1} w_1 \notin W$  for some  $N > 0$ . But in this case there can be no way to express  $t^N w_1$  as an  $A$ -linear combination of elements from  $F^r(W)$ . So  $W$  could not be a unit  $A[F^r]$ -module. Thus  $w_1$  (and likewise every other element from  $\{w_i\}$ ) must be contained in  $W$ .

Let  $V \subseteq W$  be the  $\mathbb{F}_{p^r}$ -vector space spanned by  $\{w_1, \dots, w_n\}$ . Choose an  $\mathbb{F}_{p^r}$ -basis  $\{w'_1, \dots, w'_m\}$  for the subspace  $V \cap tW$ . Each element  $w'_i$  satisfies  $t^{-1}w'_i \in W$ . Since  $W$  is closed under action by Frobenius, this implies  $t^{-p^r N} w'_i \in W$  for any  $N > 0$ . Extend  $\{w'_1, \dots, w'_m\}$  to a basis  $\{w'_1, \dots, w'_n\}$  for the entire space  $V$ . This basis determines isomorphism (3.1.6).  $\square$

**Proposition 3.1.7.** *Let  $W$  be an object from  $\mathbf{LMod}^{fu}(A[F^r])$  which is a torsion-free  $A$ -module. Then there exists a finite integral extension*

$$(3.1.8) \quad \begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ K & \longrightarrow & K', \end{array}$$

and a left  $A'[F^r]$ -module isomorphism

$$(3.1.9) \quad A' \otimes_A W \cong (K')^{\oplus m} \oplus (A')^{\oplus n-m}$$

with  $0 \leq m \leq n$ .

*Proof.* Choose a field extension  $K'/K$  according to Corollary 3.1.3 so that  $K' \otimes_A W$  is isomorphic to  $(K')^{\oplus n}$  for some  $n$ . Let  $A'$  be the integral closure of  $A$  inside of  $K'$ . Note that the Heneselian DVRs  $A$  and  $A'$  are in fact isomorphic. Thus Proposition 3.1.5 can be translated into a statement about modules over  $A'$ :

- If  $W'$  is any torsion-free unit  $A'[F^r]$ -module such that  $K' \otimes W' \cong (K')^{\oplus n}$ , then  $W' \cong K'^{\oplus m} \oplus A'^{\oplus n-m}$  for some  $m$ .

The proposition follows once we let  $W' = A' \otimes_A W$ .  $\square$

**Proposition 3.1.10.** *Let  $W$  be a free finite-rank  $A$ -module which has a unit  $A[F^r]$ -module structure. Then  $W$  is isomorphic as a left  $A[F^r]$ -module to  $A^{\oplus n}$  for some  $n$ .*



*Proof.* Consider the quotient  $W/tW$ , which is a finite-dimensional unit  $k[F^r]$ -module. By Proposition 3.0.12, this vector space has an  $F^r$ -invariant basis. To prove Proposition 3.1.10, it suffices to show that this basis can be lifted to an  $F^r$ -invariant  $A$ -module basis for  $W$ . This is accomplished by the following lemma:

**Lemma 3.1.11.** *Any  $F^r$ -invariant element of  $W/tW$  can be uniquely lifted to an  $F^r$ -invariant element of  $W$ .*

*Proof.* The lemma may be formulated in terms of commutative algebra. Let  $\{w_1, \dots, w_n\}$  be any  $A$ -module basis for  $W$ . Since  $W$  is a unit  $A[F^r]$ -module, the set  $\{F^r(w_1), \dots, F^r(w_n)\}$  is another basis, and there exists an invertible  $A$ -matrix  $(c_{ij})$  such that

$$(3.1.12) \quad w_i = \sum_{j=1}^n c_{ij} F^r(w_j).$$

An element

$$(3.1.13) \quad \sum_{k=1}^n a_k w_k \in W \quad (a_k \in A)$$

is  $F^r$ -invariant if and only if

$$(3.1.14) \quad \sum_{k=1}^n a_k^{p^r} F^r(w_k) = \sum_{k=1}^n a_k w_k = \sum_{k=1}^n a_k \sum_{j=1}^n c_{kj} F^r(w_j),$$

or equivalently,

$$(3.1.15) \quad a_k^{p^r} = \sum_{\ell=1}^n a_\ell c_{\ell k}$$

for each  $k = 1, 2, \dots, n$ . Let

$$(3.1.16) \quad R = A[X_1, \dots, X_n] / \left( \left\{ X_k^{p^r} - \sum_{\ell=1}^n X_\ell c_{\ell k} \right\}_{k=1}^n \right).$$

Then  $F^r$ -invariant elements of  $W$  may be specified by  $A$ -homomorphisms from  $R$  into  $A$ , while  $F^r$ -invariant elements of  $W/tW$  may be specified by  $A$ -homomorphisms from  $R$  into  $k$ . The claim made in the lemma, then, is equivalent to the assertion that every element of  $\text{Hom}_A(R, k)$  can be lifted to an element of  $\text{Hom}_A(R, A)$ .

This assertion becomes evident once we understand the structure of  $R$ . The extension  $A \rightarrow R$  is finite, flat, and unramified (as the reader may check), and therefore étale. Since  $A$  is a Henselian local ring,  $R$  is simply a finite direct sum of copies of  $A$ .  $\square$

Now we may complete the proof of Proposition 3.1.10. Choose an  $F^r$ -invariant  $k$ -basis for  $W/tW$ . There is a unique  $F^r$ -invariant lifting of this set to  $W$ , and by Nakayama's lemma this lifting is an  $A$ -module basis.  $\square$

**3.2. The structure of a unit  $A[F^r]$ -module.** If  $W$  is a torsion-free unit  $A[F^r]$ -module, let

$$(3.2.1) \quad W^{vec} = \bigcap_{n=0}^{\infty} t^n W \subseteq W.$$

The set  $W^{vec}$  is the largest  $K$ -vector space contained inside of  $W$ . (This definition extends naturally to modules over finite integral extensions  $A \hookrightarrow A'$  as well.) It is

easily checked that  $W^{vec}$  is stabilized by the action of  $F^r$ . Thus there is an exact sequence of left  $A[F^r]$ -modules,

$$(3.2.2) \quad 0 \rightarrow W^{vec} \rightarrow W \rightarrow W/W^{vec} \rightarrow 0.$$

This exact sequence will be the basis for the proof of the local Riemann-Hilbert correspondence.

The reader may verify the following elementary assertions for any torsion-free unit  $A[F^r]$ -module  $W$ :

- (1) The quotient  $W/W^{vec}$  inherits the structure of a unit  $A[F^r]$ -module, and  $W^{vec}$  inherits the structure of a unit  $K[F^r]$ -module.
- (2) If  $V$  is any  $K$ -vector subspace of  $W^{vec}$ , then  $(W/V)^{vec} = W^{vec}/V$ .
- (3) If  $A \hookrightarrow A'$  is any finite integral extension, then  $(A' \otimes_A W)^{vec} = A' \otimes_A W^{vec}$ .

**Proposition 3.2.3.** *Let  $W$  be an object from  $\mathbf{LMod}^{fu}(A[F^r])$  which is a torsion-free  $A$ -module. Then  $W^{vec}$  is a finite-dimensional  $K$ -vector space, and  $W/W^{vec}$  is isomorphic as a left  $A[F^r]$ -module to  $A^{\oplus n}$  for some  $n$ .*

*Proof.* The  $K$ -vector space  $W \otimes_A K$  is finite-dimensional (by Proposition 2.2.10), and  $W^{vec}$  is a  $K$ -subspace of  $W \otimes_A K$ . The first assertion follows. Let  $U = W/W^{vec}$ . Then  $U$  is a torsion-free unit  $A[F^r]$ -module such that  $U^{vec} = \{0\}$ . By Proposition 3.1.10, the proof will be completed if we can show that  $U$  is finitely-generated as an  $A$ -module.

Choose a finite integral extension  $(A', K')$  of  $(A, K)$  according to Proposition 3.1.7 so that

$$(3.2.4) \quad A' \otimes_A U \cong (K')^{\oplus m} \oplus (A')^{\oplus n-m}.$$

Since  $(A' \otimes_A U)^{vec} = A' \otimes_A U^{vec} = \{0\}$ , we must have  $m = 0$  above. Isomorphism (3.2.4) makes  $U$  isomorphic to an  $A$ -submodule of  $(A')^{\oplus n}$ . Therefore  $U$  is a finitely-generated  $A$ -module.  $\square$

The following corollary includes a converse to Proposition 3.2.3. The proof is easy and is left to the reader.

**Corollary 3.2.5.** *Let  $Y$  be a left  $A[F^r]$ -module. Then the following conditions are equivalent:*

- (1) *The module  $Y$  is a finitely-generated unit  $A[F^r]$ -module that has no  $A$ -torsion.*
- (2) *There exists an exact sequence of left  $A[F^r]$ -modules,*

$$(3.2.6) \quad 0 \rightarrow Y' \rightarrow Y \rightarrow A^{\oplus n} \rightarrow 0,$$

*where  $Y'$  is a finitely-generated unit  $K[F^r]$ -module.*  $\square$

**3.3. The local Riemann-Hilbert correspondence.** It is helpful at this point to introduce some geometric notation. Let  $Z = \text{Spec } A$ . Let  $s$  be the closed point of  $Z$  (which has residue field  $k$ ), and let  $\eta$  be the generic point of  $Z$  (which has residue field  $K$ ). Let

$$(3.3.1) \quad \bar{\eta}: \text{Spec } K^{sep} \rightarrow Z$$

be a geometric point at  $\eta$ . (Here  $K^{sep}$  denotes a separable closure of  $K$ .)

Let  $V$  be a constructible sheaf of  $\mathbb{F}_{p^r}$ -vector spaces on the scheme  $\{\eta\} \subseteq Z$ . Since  $V$  is constructible, its stalk  $M_{\bar{\eta}}$  is finite. Let

$$(3.3.2) \quad \mathcal{V}' = \mathcal{H}om_{\mathbb{F}_{p^r}}(V, \mathcal{O}_{\{\eta\}}).$$

Galois descent implies that  $\mathcal{V}'$  is a quasi-coherent  $\mathcal{O}_{\{\eta\}}$ -module. Moreover, the  $r$ th Frobenius endomorphism of  $\mathcal{O}_{\{\eta\}}$  determines a left  $\mathcal{O}_{F^r, \{\eta\}}$ -structure on  $\mathcal{V}'$  which makes  $\mathcal{V}'$  a finitely-generated unit  $\mathcal{O}_{F^r, \{\eta\}}$ -module.

At the same time, if  $\mathcal{V}$  is a finitely-generated unit  $\mathcal{O}_{F^r, \{\eta\}}$ -module, then

$$(3.3.3) \quad V' = \mathcal{H}om_{\mathcal{O}_{F^r, \{\eta\}}}(\mathcal{V}, \mathcal{O}_{\{\eta\}})$$

is a sheaf of  $\mathbb{F}_{p^r}$ -vector spaces on  $\{\eta\}$ . The stalk of  $V'$  at  $\bar{\eta}$  is

$$(3.3.4) \quad V'_{\bar{\eta}} = \text{Hom}_{K^{sep}}(\mathcal{V}_{\bar{\eta}}, K^{sep}),$$

which is made isomorphic to  $\mathbb{F}_{p^r}^n$  for some  $n$  by Proposition 3.1.1. Thus  $V'$  is a constructible  $\mathbb{F}_{p^r}$ -étale sheaf. There is a natural double-dual homomorphism

$$(3.3.5) \quad \mathcal{V} \rightarrow \mathcal{H}om_{\mathbb{F}_{p^r}}(\mathcal{H}om_{\mathcal{O}_{F^r, \{\eta\}}}(\mathcal{V}, \mathcal{O}_{\{\eta\}}), \mathcal{O}_{\{\eta\}})$$

which is easily seen to be an isomorphism by computing stalks at  $\bar{\eta}$ . Likewise, the double-dual homomorphism

$$(3.3.6) \quad V \rightarrow \mathcal{H}om_{\mathcal{O}_{F^r, \{\eta\}}}(\mathcal{H}om_{\mathbb{F}_{p^r}}(V, \mathcal{O}_{\{\eta\}}), \mathcal{O}_{\{\eta\}})$$

is an isomorphism.

Let  $\mathbf{Mod}^c(\{\eta\}, \mathbb{F}_{p^r})$  be the full subcategory of constructible sheaves in  $\mathbf{Mod}(\{\eta\}, \mathbb{F}_{p^r})$ . The functors  $\mathcal{H}om_{\mathbb{F}_{p^r}}(\cdot, \mathcal{O}_{\{\eta\}})$  and  $\mathcal{H}om_{\mathcal{O}_{F^r, \{\eta\}}}(\cdot, \mathcal{O}_{\{\eta\}})$  determine an equivalence of categories between  $\mathbf{LMod}^{fu}(\{\eta\}, \mathcal{O}_{F^r, \{\eta\}})$  and  $\mathbf{Mod}^c(\{\eta\}, \mathbb{F}_{p^r})$ . The local Riemann-Hilbert correspondence simply extends this equivalence to the scheme  $Z$ .

**Theorem 3.3.7.** *Let  $M$  be a constructible  $\mathbb{F}_{p^r}$ -étale sheaf on  $Z$  whose sections all have open support. Then the sheaf*

$$(3.3.8) \quad \mathcal{M}' = \mathcal{H}om_{\mathbb{F}_{p^r}}(M, \mathcal{O}_Z)$$

*is a finitely-generated unit  $\mathcal{O}_{F^r, Z}$ -module. The double-dual homomorphism*

$$(3.3.9) \quad M \rightarrow \mathcal{H}om_{\mathcal{O}_{F^r, Z}}(\mathcal{M}', \mathcal{O}_Z)$$

*is an isomorphism.*

*Let  $\mathcal{M}$  be a sheaf from  $\mathbf{LMod}^{fu}(Z, \mathcal{O}_{F^r, Z})$  which has a torsion-free  $\mathcal{O}_Z$ -module structure. Then the sheaf*

$$(3.3.10) \quad \mathcal{M}' = \mathcal{H}om_{\mathcal{O}_{F^r, Z}}(\mathcal{M}, \mathcal{O}_Z)$$

*is a constructible sheaf of  $\mathbb{F}_{p^r}$ -vector spaces. The double-dual homomorphism*

$$(3.3.11) \quad \mathcal{M} \rightarrow \mathcal{H}om_{\mathcal{O}_{F^r, Z}}(\mathcal{M}', \mathcal{O}_Z)$$

*is an isomorphism.*

*Proof.* For any torsion-free  $\mathcal{N} \in \mathbf{Ob} \mathbf{LMod}^{fu}(Z, \mathcal{O}_{F^r, Z})$  let  $\mathcal{N}^{vec}$  denote the subsheaf generated by  $\Gamma(Z, \mathcal{N})^{vec}$  (see Section 3.2). For any  $N \in \mathbf{Ob} \mathbf{Mod}^c(Z, \mathbb{F}_{p^r})$ , let  $N^{con} \subseteq N$  denote the subsheaf generated by the global sections of  $N$ . The reader may check the following observations:

- (1) For any torsion-free finitely-generated unit  $\mathcal{O}_{F^r, Z}$ -module  $\mathcal{N}$ , the sheaf

$$(3.3.12) \quad \mathcal{H}om_{\mathcal{O}_{F^r, Z}}(\mathcal{N}, \mathcal{O}_Z)^{con}$$

is the sheaf of  $\mathcal{O}_{F^r, Z}$ -homomorphisms from  $\mathcal{N}$  into  $\mathcal{O}_Z$  that kill  $\mathcal{N}^{vec}$ .

(2) For any constructible  $\mathbb{F}_{p^r}$ -étale sheaf  $N$  on  $Z$ , the sheaf

$$(3.3.13) \quad \mathcal{H}om_{\mathbb{F}_{p^r}}(N, \mathcal{O}_Z)^{vec}$$

is the sheaf of  $\mathbb{F}_{p^r}$ -homomorphisms from  $N$  into  $\mathcal{O}_Z$  that kill  $N^{con}$ .

This symmetry has a number of useful consequences. The sheaf  $\mathcal{M}'^{vec}$  is isomorphic to

$$(3.3.14) \quad \mathcal{H}om_{\mathcal{O}_{F^r, Z}}(M/M^{con}, \mathcal{O}_Z).$$

The quotient  $M/M^{con}$  is simply the pushforward of an étale sheaf on  $\{\eta\}$ . The Riemann-Hilbert correspondence over  $\{\eta\}$  implies that  $\mathcal{M}'^{vec}$  is a finitely-generated unit  $\mathcal{O}_{F^r, Z}$ -module. Meanwhile, the quotient sheaf  $\mathcal{M}'/\mathcal{M}'^{vec}$  is isomorphic to

$$(3.3.15) \quad \mathcal{H}om_{\mathbb{F}_{p^r}}(M^{con}, \mathcal{O}_Z).$$

Since  $M^{con}$  is a constant sheaf, this sheaf is simply isomorphic to  $\mathcal{O}_Z^{\oplus d}$  for some  $d$ . Thus there is an exact sequence

$$(3.3.16) \quad 0 \rightarrow \mathcal{M}'^{vec} \rightarrow \mathcal{M}' \rightarrow \mathcal{O}_Z^{\oplus d} \rightarrow 0,$$

which implies (by Corollary 3.2.5) that  $\mathcal{M}'$  is a finitely-generated unit  $\mathcal{O}_{F^r, Z}$ -module.

Let

$$(3.3.17) \quad M'' = \mathcal{H}om_{\mathcal{O}_{F^r, Z}}(\mathcal{M}', \mathcal{O}_Z)$$

be the double-dual of  $M$ . The symmetry discussed above makes the sheaf  $M''^{con}$  naturally isomorphic to the double-dual of the sheaf  $M^{con}$ , and makes  $M''/M''^{con}$  naturally isomorphic to the double-dual of the sheaf  $M/M^{con}$ . There are double-dual maps

$$(3.3.18) \quad M^{con} \rightarrow M''^{con} \text{ and } M/M^{con} \rightarrow M''/M''^{con}.$$

It is easily seen that  $M^{con}$  is isomorphic to its double-dual. The same is true for  $M/M^{con}$  by the Riemann-Hilbert correspondence over  $\{\eta\}$ . Thus in the diagram

$$(3.3.19) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M^{con} & \longrightarrow & M & \longrightarrow & M/M^{con} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M''^{con} & \longrightarrow & M'' & \longrightarrow & M''/M''^{con} \longrightarrow 0, \end{array}$$

both of the outside vertical maps are isomorphisms. The homomorphism  $M \rightarrow M''$  must be an isomorphism by the 5-lemma.

The proof of the second part of Theorem 3.3.7 proceeds similarly. One needs only the additional fact that  $\mathcal{M}/\mathcal{M}^{vec}$  is isomorphic to  $\mathcal{O}_Z^{\oplus e}$  for some  $e > 0$ . (This is implied by Proposition 3.2.3.)  $\square$

**3.4. Roots of unit  $A[F^r]$ -modules.** We revert to algebraic notation. Let  $W$  be a finitely-generated unit  $A[F^r]$ -module which has no  $A$ -torsion. Then an  $A$ -submodule  $W_0 \subseteq W$  is a root if:

- (1)  $W_0$  is a finitely-generated  $A$ -module,
- (2) the  $A$ -submodule generated by  $F^r(W_0) \subseteq W$  contains  $W_0$ , and
- (3)  $W$  is generated as a left  $A[F^r]$ -module by  $W_0$ .

As in Proposition 2.2.3, a root determines a filtration

$$(3.4.1) \quad W_0 \subseteq W_1 \subseteq W_2 \subseteq \dots$$

for  $W$ , in which  $W_i$  is the  $A$ -submodule of  $W$  generated by  $F^{ri}(W_0)$ . Each of these modules has an isomorphism  $W_i \cong F_A^{ri*} W_0$  given by the structural morphism of  $W$ . (Here  $F_A^{ri*} W_0$  denotes the tensor product  $A \otimes_A W_0$  taken via  $F_A^{ri}: A \rightarrow A$ .) Each module  $W_i$  is a free  $A$ -module with rank equal to  $\dim_K K \otimes_A W$ .

Our goal in this subsection is to establish a useful exact sequence that involves the dual of a root.

Suppose that  $W_0$  is a root for  $W$ . Let  $\{w_1, \dots, w_n\}$  be an  $A$ -module basis for  $W_0$ . By property (2) above, each element of the basis may be expressed (uniquely, in fact) as

$$(3.4.2) \quad w_i = \sum_{j=1}^n a_{ij} F^r(w_j),$$

with  $a_{ij} \in A$ .

Let

$$(3.4.3) \quad W_0^\vee = \text{Hom}_A(W_0, A)$$

be the  $A$ -module dual of  $W_0$ . The module  $W_0^\vee$  has a canonical left  $A$ -module structure: if  $\phi: W \rightarrow A$  is any  $A$ -module homomorphism, we define  $F^r(\phi)$  to be the composition of the diagram

$$(3.4.4) \quad W_0 \longrightarrow W_1 \xrightarrow{\cong} F_A^{r*} W_0 \xrightarrow{(a \otimes w) \mapsto a F^r(\phi(w))} A.$$

The left  $A[F^r]$ -module structure of  $W_0^\vee$  may also be understood in terms of the basis  $\{w_1, \dots, w_n\}$  chosen above. Let  $\{w_1^\vee, \dots, w_n^\vee\}$  be the dual of this basis. Then (as the reader may check),

$$(3.4.5) \quad F^r(w_i^\vee) = \sum_{j=1}^n a_{ji} w_j^\vee.$$

Suppose that  $\psi$  is an element of  $W_0^\vee$  which is invariant under the action of  $F^r$ . Then, the composite map

$$(3.4.6) \quad W_1 \xrightarrow{\cong} F_A^{r*} W_0 \xrightarrow{(a \otimes w) \mapsto a F^r(\psi(w))} A$$

is compatible with the map  $\psi: W_0 \rightarrow A$  itself. In fact, there is a sequence of induced maps

$$(3.4.7) \quad W_i \xrightarrow{\cong} F_A^{ri*} W_0 \longrightarrow A.$$

(for  $i = 1, 2, \dots$ ) all of which are compatible via restriction. Taken together, these maps determine a left  $A[F^r]$ -module homomorphism from  $W$  into  $A$ . In this way we see that the  $F^r$ -invariant elements of  $W_0^\vee$  are precisely the elements that arise by restriction of maps from  $\text{Hom}_{A[F^r]}(W, A)$ .

**Proposition 3.4.8.** *Let  $W$  be a finitely-generated unit  $A[F^r]$ -module that has no  $A$ -torsion. Suppose that  $W_0 \subseteq W$  is a root of  $W$ . Then restriction determines a bijection*

$$(3.4.9) \quad \text{Hom}_{A[F^r]}(W, A) \rightarrow (W_0^\vee)^{F^r}.$$

This proposition has the following consequence.

**Theorem 3.4.10.** *Let  $W$  be an object from  $\mathbf{LMod}^{fu}(A[F^r])$  which is a torsion-free  $A$ -module. Suppose that  $W$  has a root  $W_0$ . Let  $M = \mathrm{Hom}_{A[F^r]}(W, A)$ . Then the sequence*

$$(3.4.11) \quad 0 \longrightarrow M \longrightarrow W_0^\vee \xrightarrow{(1-F^r)} W_0^\vee \longrightarrow 0$$

*is exact.*

*Proof.* We need only to show that the action of  $1 - F^r$  on  $W_0^\vee$  is surjective. This is accomplished by the following lemma.

**Lemma 3.4.12.** *For any  $v \in W_0^\vee$ , there exists an element  $v' \in W_0^\vee$  such that  $v' - F^r(v') = v$ .*

*Proof.* Using the notation from earlier in this subsection, we may write

$$(3.4.13) \quad v = \sum_{i=1}^n b_i w_i^\vee$$

with  $b_i \in A$ . Solving the equation  $v' - F^r(v') = v$  amounts to finding elements  $b'_1, \dots, b'_n \in A$  such that

$$(3.4.14) \quad \sum_{i=1}^n b'_i w_i^\vee - F^r \left( \sum_{i=1}^n b'_i w_i^\vee \right) = \sum_{i=1}^n b_i w_i^\vee,$$

which is equivalent to solving the system of equations

$$(3.4.15) \quad b'_i - \sum_{j=1}^n (b'_j)^{p^r} a_{ji} = b_i$$

( $i = 1, 2, \dots, n$ ). This in turn is equivalent to finding a homomorphism of the  $A$ -algebra

$$(3.4.16) \quad S = A[Y_1, \dots, Y_n] / \left( \left\{ Y_i - \sum_{j=1}^n Y_j^{p^r} a_{ji} - b_i \right\}_{i=1}^n \right)$$

into  $A$ . By calculating the module of relative differentials of  $S$  over  $A$  one sees that  $S$  is a finite étale  $A$ -algebra, and is therefore simply isomorphic to a direct sum of copies of  $A$ . Thus sections  $S \rightarrow A$  clearly exist, and the lemma is proved.  $\square$

Proposition 3.4.8 and Lemma 3.4.12 together prove Theorem 3.4.10.  $\square$

The case of Theorem 3.4.10 that is of interest is when  $W$  and  $M$  are stalks of sheaves related by the Riemann-Hilbert correspondence. This case will come up in Section 4.

**3.5. The minimal root index.** The concept of a root leads to the definition of an invariant which measures the complexity of objects from  $\mathbf{LMod}^{fu}(A[F^r])$ . The basis for the invariant is the following theorem, which is a special case of a result of M. Blickle:

**Theorem 3.5.1.** *Let  $W$  be finitely-generated unit  $A[F^r]$ -module. Then  $W$  has a unique root  $W_0$  which is contained in every other root.*

*Proof.* See Theorem 2.10 from [1].  $\square$

**Definition 3.5.2.** Let  $W$  be an object from  $\mathbf{LMod}^{fu}(A[F^r])$  which is a torsion-free  $A$ -module. Let  $W_0$  be the minimal root in  $W$ , and let  $W_1 \subseteq W$  be the  $A$ -submodule generated by  $F^r(W_0)$ . The minimal root index of  $W$  is

$$(3.5.3) \quad \frac{\dim_k(W_1/W_0)}{p^r - 1}.$$

Let  $M$  be a constructible  $\mathbb{F}_{p^r}$ -étale sheaf on  $\mathrm{Spec} A$ . Then the minimal root index of  $M$  is the minimal root index of

$$(3.5.4) \quad \mathrm{Hom}_{\mathbb{F}_{p^r}}(M, \mathcal{O}_{\mathrm{Spec} A})$$

(the dual of  $M$  under the Riemann-Hilbert correspondence).

Note that the minimal root index is always finite. (Since  $W_1$  and  $W_0$  have the same rank as  $A$ -modules,  $W_1/W_0$  is a torsion  $A$ -module, and is therefore finite-dimensional over  $k$ .) However it is not necessarily integral, as the following example calculation shows.

**Example 3.5.5.** Let  $W$  be a  $K$ -vector space generated by a single element  $e$ , and define the left  $A[F^r]$ -module structure on  $W$  by

$$(3.5.6) \quad F^r(e) = t^{(p^r-1)/2}e.$$

Finitely-generated  $A$ -submodules of  $W$  are all of the form  $A(t^N e)$ , with  $N \in \mathbb{Z}$ . The smallest such module which satisfies the root properties is  $A(t^{-1}e)$ . In this case

$$(3.5.7) \quad W_0 = A(t^{-1}e) \text{ and } W_1 = A(t^{(-p^r-1)/2}e).$$

The  $k$ -dimension of  $W_1/W_0$  is  $(p^r - 1)/2$ , and the minimal root index of  $W$  is  $\frac{1}{2}$ .

Note that in this case, the Riemann-Hilbert dual of  $W$  is an  $\mathbb{F}_{p^r}$ -étale sheaf on  $\mathrm{Spec} A$  of generic rank 1. The dual of  $W$  is a functor which associates to any étale ring extension  $A \rightarrow B$  the vector space  $\mathrm{Hom}_{B[F^r]}(B \otimes W, B)$ . This sheaf has nontrivial sections, for example, for the  $k$ -algebra homomorphism  $A \rightarrow K$  which maps  $t$  to  $t^2$ .

The next proposition follows easily from Proposition 3.1.10. The proof is left to the reader.

**Proposition 3.5.8.** Let  $W$  be an object from  $\mathbf{LMod}^{fu}(A[F^r])$  which is a free finite-rank  $A$ -module. Then the minimal root for  $W$  is  $W$  itself. The minimal root index for  $W$  is zero.

#### 4. THE RIEMANN-HILBERT CORRESPONDENCE ON A CURVE

Throughout this section, let  $X$  be a smooth  $k$ -curve. The Riemann-Hilbert correspondence over  $X$  relates  $\mathbb{F}_{p^r}$ -étale sheaves on  $X$  to unit  $\mathcal{O}_{F^r, X}$ -modules. Some relationships between étale cohomology and coherent cohomology can be deduced from the correspondence. In this section we will develop the Riemann-Hilbert correspondence over  $X$  by building on the local results from Section 3.

For any smooth  $k$ -scheme  $Z$ , let  $\mathbf{Mod}^c(Z, \mathbb{F}_{p^r})$  denote the category of constructible  $\mathbb{F}_{p^r}$ -étale sheaves on  $Z$ . If  $z$  is a closed point of  $Z$ , let  $\mathcal{O}_{Z, z}$  denote the stalk of the étale coordinate sheaf of  $Z$ . If  $Q$  is an étale sheaf on  $Z$ , let  $Q_{(z)}$  denote the pullback of  $Q$  via the natural morphism

$$(4.0.9) \quad \mathrm{Spec} \mathcal{O}_{Z, z} \rightarrow Z.$$

**4.1. The functor  $\mathcal{H}om_{\mathbb{F}_{p^r}}(\cdot, \mathcal{O}_X)$ .** If  $M$  is an  $\mathbb{F}_{p^r}$ -étale sheaf on  $X$ , then the sheaf  $\mathcal{H}om_{\mathbb{F}_{p^r}}(M, \mathcal{O}_X)$  has a left  $\mathcal{O}_{F^r, X}$ -module structure given by the left  $\mathcal{O}_{F^r, X}$ -module structure of  $\mathcal{O}_X$ . The next proposition shows that the functor  $\mathcal{H}om_{\mathbb{F}_{p^r}}(\cdot, \mathcal{O}_X)$  is compatible with the analogous functor from the local Riemann-Hilbert correspondence (see Theorem 3.3.7).

**Proposition 4.1.1.** *Let  $M$  be an object of  $\mathbf{Mod}^{\mathcal{F}}(X, \mathbb{F}_{p^r})$ . Let  $x$  be a closed point of  $X$ . The natural homomorphism*

$$(4.1.2) \quad \mathcal{H}om_{\mathbb{F}_{p^r}}(M, \mathcal{O}_X)_x \rightarrow \mathrm{Hom}_{\mathbb{F}_{p^r}}(M_{(x)}, \mathcal{O}_{\mathrm{Spec} \mathcal{O}_{X,x}})$$

*is an isomorphism.*

We will prove Proposition 4.1.1 by reduction to the following special case.

**Proposition 4.1.3.** *Let  $Y$  be a smooth affine curve over  $k$ . Let  $Z \rightarrow Y$  be a finite Galois cover which is totally ramified at one closed point  $y \in |Y|$  and unramified elsewhere. Let  $\{z\}$  be the pre-image of  $\{y\}$  in  $Z$ , and let  $Z' \subseteq Z$  be the complement of  $\{z\}$ . Let  $N$  be a constructible  $\mathbb{F}_{p^r}$ -étale sheaf on  $Y$  such that  $N|_{Z'}$  is constant. Then the natural homomorphism*

$$(4.1.4) \quad \mathcal{H}om_{\mathbb{F}_{p^r}}(N, \mathcal{O}_Y)_y \rightarrow \mathrm{Hom}_{\mathbb{F}_{p^r}}(N_{(y)}, \mathcal{O}_{\mathrm{Spec} \mathcal{O}_{Y,y}})$$

*is an isomorphism.*

*Proof.* The curves  $Y$ ,  $Z$ , and  $Z'$  are all affine. Let  $Y = \mathrm{Spec} R$ ,  $Z = \mathrm{Spec} S$ , and  $Z' = \mathrm{Spec} S'$ . If  $\mathrm{Spec} Q \rightarrow \mathrm{Spec} R$  is any étale morphism, then morphisms

$$(4.1.5) \quad N|_{\mathrm{Spec} Q} \rightarrow \mathcal{O}_{\mathrm{Spec} Q}$$

may be expressed as commutative diagrams

$$(4.1.6) \quad \begin{array}{ccc} N(\mathrm{Spec} S') & \xrightarrow{\phi} & S' \otimes_R Q \\ \rho_1 \uparrow & & \uparrow \rho_2 \\ N(\mathrm{Spec} R) & \xrightarrow{\psi} & Q \end{array}$$

in which  $\rho_1$  and  $\rho_2$  are sheaf restriction maps,  $\phi$  and  $\psi$  are  $\mathbb{F}_{p^r}$ -linear homomorphisms, and  $\phi$  is  $\mathrm{Aut}(S'/R)$ -equivariant. Similarly, morphisms

$$(4.1.7) \quad N_{(y)} \rightarrow \mathcal{O}_{\mathrm{Spec} \mathcal{O}_{Y,y}}$$

may be expressed as commutative diagrams

$$(4.1.8) \quad \begin{array}{ccc} N(\mathrm{Spec} S') & \longrightarrow & S' \otimes_R \mathcal{O}_{Y,y} \\ \uparrow & & \uparrow \\ N(\mathrm{Spec} R) & \longrightarrow & \mathcal{O}_{Y,y} \end{array}$$

in which the vertical maps are sheaf restriction maps, the horizontal maps are  $\mathbb{F}_{p^r}$ -linear homomorphisms, and the top map is  $\mathrm{Aut}(S'/R)$ -equivariant.

Suppose that

$$(4.1.9) \quad \begin{array}{ccc} N(\mathrm{Spec} S') & \longrightarrow & S' \otimes_R \mathcal{O}_{Y,y} \\ \uparrow & & \uparrow \\ N(\mathrm{Spec} R) & \longrightarrow & \mathcal{O}_{Y,y} \end{array}$$



is the diagram for a morphism  $N_{(y)} \rightarrow \mathcal{O}_{\text{Spec } \mathcal{O}_{Y,y}}$ . Since  $N(\text{Spec } R)$  and  $N(\text{Spec } S')$  are finite, there exists an étale  $R$ -algebra  $P \subseteq \mathcal{O}_{Y,y}$  such that the images of  $N(\text{Spec } R)$  and  $N(\text{Spec } S')$  are contained in  $P$  and  $S' \otimes_R P$ , respectively. Thus (4.1.9) determines a commutative diagram

$$(4.1.10) \quad \begin{array}{ccc} N(\text{Spec } S') & \longrightarrow & S' \otimes_R P \\ \uparrow & & \uparrow \\ N(\text{Spec } R) & \longrightarrow & P. \end{array}$$

We conclude that any  $\mathbb{F}_{p^r}$ -linear morphism  $N_{(y)} \rightarrow \mathcal{O}_{\text{Spec } \mathcal{O}_{Y,y}}$  extends to an  $\mathbb{F}_{p^r}$ -linear morphism from  $N \rightarrow \mathcal{O}_Y$  on some étale neighborhood of  $y$ . We have constructed an inverse to homomorphism (4.1.4).  $\square$

*Proof of Proposition 4.1.1.* Replacing  $X$  with an open subcurve if necessary, we may assume that  $X$  is affine and that the constructible sheaf  $M$  is locally constant on  $X \setminus \{x\}$ . Let  $Z' \rightarrow X \setminus \{x\}$  be a finite Galois étale cover such that  $M|_{Z'}$  is constant.

Let  $K_0/K(X)$  be the largest subextension of  $K(Z')/K(X)$  that is unramified at  $x$ . The tower of field extensions

$$(4.1.11) \quad K(Z') \supseteq K_0 \supseteq K(X)$$

determines a diagram of smooth projective curves,

$$(4.1.12) \quad \begin{array}{ccc} \overline{Z'} & \longrightarrow & W \\ & \searrow & \downarrow \\ & & \overline{X} \end{array}$$

where  $\overline{Z'}$  and  $\overline{X}$  are the smooth projective closures of  $Z'$  and  $X$ , respectively, and  $W$  is the unique smooth projective curve over  $k$  whose fraction field is  $K_0$ . Let

$$(4.1.13) \quad \begin{array}{ccc} \overline{Z'} \times_{\overline{X}} X & \longrightarrow & W \times_{\overline{X}} X \\ & \searrow & \downarrow \\ & & X \end{array}$$

be the diagram obtained from (4.1.12) via base change. The morphism

$$(4.1.14) \quad \overline{Z'} \times_{\overline{X}} X \rightarrow X$$

is Galois and finite, and étale away from  $x$ . The morphism

$$(4.1.15) \quad W \times_{\overline{X}} X \rightarrow X$$

is Galois and finite and étale at all points of  $X$ . The morphism

$$(4.1.16) \quad \overline{Z'} \times_{\overline{X}} X \rightarrow W \times_{\overline{X}} X$$

is Galois and finite, étale away from  $x$ , and totally ramified at  $x$ . Proposition 4.1.3 may therefore be applied with  $Y = W \times_{\overline{X}} X$ ,  $Z = \overline{Z'} \times_{\overline{X}} X$ , and  $N = M|_{W \times_{\overline{X}} X}$ . This application completes the current proof, since the assertion of Proposition 4.1.1 is local at  $x$ .  $\square$

**Proposition 4.1.17.** *Let  $M$  be a constructible  $\mathbb{F}_{p^r}$ -étale sheaf on  $X$  whose sections all have open support. Let*

$$(4.1.18) \quad \mathcal{M} = \mathcal{H}om_{\mathbb{F}_{p^r}}(M, \mathcal{O}_X),$$

*with a left  $\mathcal{O}_{F^r, X}$ -module structure given by the Frobenius endomorphism of  $\mathcal{O}_X$ . Then  $\mathcal{M}$  is an lfgu (locally finitely-generated unit)  $\mathcal{O}_{F^r, X}$ -module, and it is torsion-free as an  $\mathcal{O}_X$ -module.*

*Proof.* The proof of this proposition consists of three lemmas.

**Lemma 4.1.19.** *The sheaf  $\mathcal{M}$  is a quasi-coherent  $\mathcal{O}_X$ -module.*

*Proof.* Let  $U \subseteq X$  be a nonempty open subset on which  $M$  is locally constant. Let  $j: U \rightarrow X$  be the inclusion morphism. The sheaf

$$(4.1.20) \quad \mathcal{H}om_{\mathbb{F}_{p^r}}(M|_U, \mathcal{O}_U)$$

is locally free of finite rank as an  $\mathcal{O}_U$ -module. The pushforward

$$(4.1.21) \quad j_* \mathcal{H}om_{\mathbb{F}_{p^r}}(M|_U, \mathcal{O}_U) \cong \mathcal{H}om_{\mathbb{F}_{p^r}}(M, j_* \mathcal{O}_U)$$

is a quasi-coherent  $\mathcal{O}_X$ -module. There is a natural morphism

$$(4.1.22) \quad \mathcal{M} \hookrightarrow \mathcal{H}om_{\mathbb{F}_{p^r}}(M, j_* \mathcal{O}_U).$$

To show that  $\mathcal{M}$  is quasi-coherent, it suffices to show that the cokernel of this morphism is quasi-coherent.

The image of (4.1.22) consists of the morphisms  $M \rightarrow j_* \mathcal{O}_U$  that map  $M_x$  into  $\mathcal{O}_{X,x}$  for each  $x \in |X \setminus U|$ . Suppose that  $x$  is an element of  $|X \setminus U|$ , and suppose that  $\phi$  is a morphism from  $M$  to  $j_* \mathcal{O}_U$  over a Zariski open neighborhood of  $x$ . Choose a local parameter  $t$  at  $x$ . Since  $M_x$  is finite, we may choose  $n$  sufficiently large so that  $t^n \phi$  maps  $M_x$  into  $\mathcal{O}_{X,x}$ . We conclude that the cokernel of (4.1.22) is a quasi-coherent skyscraper sheaf supported at  $|X \setminus U|$ . This completes the proof.  $\square$

**Lemma 4.1.23.** *The structural morphism*

$$(4.1.24) \quad F_X^* \mathcal{M} \rightarrow \mathcal{M}$$

*is an isomorphism.*

*Proof.* It suffices to show that for any closed point  $x \in |X|$ , the structural morphism of  $\mathcal{M}_x$  is an isomorphism. By Proposition 4.1.1, there exist isomorphisms

$$(4.1.25) \quad \mathcal{M}_x \rightarrow \mathcal{H}om_{\mathbb{F}_{p^r}}(M_{(x)}, \mathcal{O}_{\text{Spec } \mathcal{O}_{X,x}})$$

for each closed point  $x \in |X|$ . By Theorem 3.3.7, each left  $\mathcal{O}_{X,x}[F^r]$ -module

$$(4.1.26) \quad \mathcal{H}om_{\mathbb{F}_{p^r}}(M_{(x)}, \mathcal{O}_{\text{Spec } \mathcal{O}_{X,x}})$$

is a unit  $\mathcal{O}_{X,x}[F^r]$ -module.  $\square$

**Lemma 4.1.27.** *The left  $\mathcal{O}_{F^r, X}$ -module  $\mathcal{M}$  is generated by a finite number of sections.*

*Proof.* Let  $U \subseteq X$  be a Zariski open subset on which  $M$  is locally constant. By Proposition 4.1.1 and Theorem 3.3.7, each stalk  $\mathcal{M}_x$  is a finitely-generated unit  $\mathcal{O}_{X,x}[F^r]$ -module. For each point  $x \in |X \setminus U|$ , choose a finite set of sections of  $\mathcal{M}$  on a Zariski open neighborhood of  $x$  which generate  $\mathcal{M}_x$  as a left  $\mathcal{O}_{X,x}[F^r]$ -module. Choose a finite set of sections of the coherent  $\mathcal{O}_U$ -module  $\mathcal{M}|_U$  which generate  $\mathcal{M}|_U$

as an  $\mathcal{O}_U$ -module. Let  $\mathcal{M}' \subseteq \mathcal{M}$  be the sub-left- $\mathcal{O}_{F^r, X}$ -module generated by all of the aforementioned sections. The stalk of  $\mathcal{M}'$  at any closed point  $x$  is equal to  $\mathcal{M}$ . Therefore  $\mathcal{M}' = \mathcal{M}$ .  $\square$

It is clear that  $\mathcal{M}$  is torsion-free as an  $\mathcal{O}_X$ -module. The proof of Proposition 4.1.17 is complete.  $\square$

**4.2. The functor  $\mathcal{H}om_{\mathcal{O}_{F^r, X}}(\cdot, \mathcal{O}_X)$ .** If  $\mathcal{M}$  is a unit  $\mathcal{O}_{F^r, X}$ -module, then the sheaf of homomorphisms

$$(4.2.1) \quad \mathcal{H}om_{\mathcal{O}_{F^r, X}}(\mathcal{M}, \mathcal{O}_X)$$

is a sheaf of  $\mathbb{F}_{p^r}$ -vector spaces on  $X$ .

**Proposition 4.2.2.** *Let  $\mathcal{M}$  be an object of  $\mathbf{LMod}^{fu}(X, \mathcal{O}_{F^r, X})$ . Let  $x$  be a closed point of  $X$ . The natural homomorphism*

$$(4.2.3) \quad \mathcal{H}om_{\mathcal{O}_{F^r, X}}(\mathcal{M}, \mathcal{O}_X)_x \rightarrow \mathrm{Hom}_{\mathcal{O}_{X, x}[F^r]}(\mathcal{M}_x, \mathcal{O}_{X, x})$$

*is an isomorphism.*

*Proof.* It is clear that (4.2.3) is injective. To prove the proposition it suffices to show that any element of

$$(4.2.4) \quad \mathrm{Hom}_{\mathcal{O}_{X, x}[F^r]}(\mathcal{M}_x, \mathcal{O}_{X, x})$$

may be extended to a left  $\mathcal{O}_{F^r, X}$ -module homomorphism from  $\mathcal{M}$  to  $\mathcal{O}_X$  over an étale neighborhood of  $x$ .

Let

$$(4.2.5) \quad \phi: \mathcal{M}_x \rightarrow \mathcal{O}_{X, x}$$

be a left  $\mathcal{O}_{X, x}[F^r]$ -module homomorphism. Let  $U \subseteq X$  be an affine neighborhood of  $x$ . Let  $R = \Gamma(U, \mathcal{O}_U)$  and  $P = \Gamma(U, \mathcal{M})$ . Let  $\{p_1, \dots, p_e\} \subseteq P$  be a subset which generates  $P$  as a left  $R[F^r]$ -module. Choose an étale  $R$ -algebra  $R' \subseteq \mathcal{O}_{X, x}$  large enough to contain the images of the stalks of each  $p_i$  under  $\phi$ . The homomorphism

$$(4.2.6) \quad \mathcal{O}_{X, x} \otimes_R P \rightarrow \mathcal{O}_{X, x}$$

determined by  $\phi$  restricts to a homomorphism

$$(4.2.7) \quad R' \otimes_R P \rightarrow R'.$$

Thus there is a homomorphism from  $\mathcal{M}|_{\mathrm{Spec} R'}$  to  $\mathcal{O}_{\mathrm{Spec} R'}$  whose stalk is  $\phi$ .  $\square$

**Proposition 4.2.8.** *Let  $\mathcal{M}$  be an object of  $\mathbf{LMod}^{fu}(X, \mathcal{O}_{F^r, X})$ . Let*

$$(4.2.9) \quad M = \mathcal{H}om_{\mathcal{O}_{F^r, X}}(\mathcal{M}, \mathcal{O}_X).$$

*Then there exists a nonempty étale  $X$ -scheme  $V$  such that  $M|_V$  is a constant  $\mathbb{F}_{p^r}$ -sheaf of finite rank.*

*Proof.* By Proposition 2.2.8, there exists a nonempty open subset  $X' \subseteq X$  such that  $\mathcal{M}|_{X'}$  is a coherent  $\mathcal{O}_{X'}$ -module. Let  $\alpha \in |X'|$  denote the generic point, and let

$$(4.2.10) \quad \bar{\alpha}: \mathrm{Spec} \overline{k(\alpha)} \rightarrow X'$$

denote a geometric point at  $\alpha$ . The geometric stalk  $\mathcal{M}_{\bar{\alpha}}$  has a  $\overline{k(\alpha)}$ -basis that is fixed by  $F^r$  (by Proposition 3.0.12). Choose an étale scheme  $U$  over  $X'$  on which there exist representatives

$$(4.2.11) \quad m_1, \dots, m_e \in \mathcal{M}(U)$$

for the elements of this basis. The coherent subsheaf of  $\mathcal{M}|_U$  generated by  $\{m_i\}_{i=1}^e$  has the same generic rank as  $\mathcal{M}|_U$ . Let  $V \subseteq U$  be a nonempty open subset on which these two sheaves are equal. Then  $\mathcal{M}|_V$  is isomorphic as a left  $\mathcal{O}_{F^r, X}$ -module to  $\mathcal{O}_V^{\oplus e}$ . Therefore  $M|_V$  is isomorphic to a constant  $\mathbb{F}_{p^r}$ -étale sheaf of rank  $e$ .  $\square$

**Corollary 4.2.12.** *Let  $\mathcal{M}$  be an object of  $\mathbf{LMod}^{fu}(X, \mathcal{O}_{F^r, X})$  which is torsion-free as an  $\mathcal{O}_X$ -module. Then*

$$(4.2.13) \quad M = \mathcal{H}om_{\mathcal{O}_{F^r, X}}(\mathcal{M}, \mathcal{O}_X)$$

*is a constructible  $\mathbb{F}_{p^r}$ -étale sheaf on  $X$ .*

*Proof.* Proposition 4.2.8 implies that  $M$  is locally constant on a nonempty open subset of  $X$ . Theorem 3.3.7 implies (via Proposition 4.2.2) that the stalks of  $M$  are finite.  $\square$

**Proposition 4.2.14.** *Let  $\mathcal{M}$  be a torsion-free lfgu  $\mathcal{O}_{F^r, X}$ -module. The double-dual homomorphism*

$$(4.2.15) \quad \mathcal{M} \rightarrow \mathcal{H}om_{\mathbb{F}_{p^r}}(\mathcal{H}om_{\mathcal{O}_{F^r, X}}(\mathcal{M}, \mathcal{O}_X), \mathcal{O}_X)$$

*is an isomorphism.*

**Proposition 4.2.16.** *Let  $M$  be a constructible  $\mathbb{F}_{p^r}$ -étale sheaf on  $X$  whose sections all have open support. The double-dual homomorphism*

$$(4.2.17) \quad M \rightarrow \mathcal{H}om_{\mathcal{O}_{F^r, X}}(\mathcal{H}om_{\mathbb{F}_{p^r}}(M, \mathcal{O}_X), \mathcal{O}_X)$$

*is an isomorphism.*

*Proof of Propositions 4.2.14 and 4.2.16.* It suffices to show that morphisms (4.2.15) and (4.2.17) induce isomorphisms on closed stalks. This assertion follows from Theorem 3.3.7 via Propositions 4.1.1 and 4.2.2.  $\square$

**4.3. Roots on curves.** The next two theorems globalize results on roots from Section 3.

**Theorem 4.3.1.** *Let  $\mathcal{M}$  be a torsion-free lfgu  $\mathcal{O}_{F^r, X}$ -module. Then  $\mathcal{M}$  has a unique minimal root  $\mathcal{M}_0$  which is contained in every other root. For any closed point  $x \in |X|$ , the stalk of  $\mathcal{M}_0$  at  $x$  is the minimal root of  $\mathcal{M}_x$ .*

*Proof.* Our method is to define a subsheaf  $\mathcal{M}_0$  and then prove that it has the desired properties. For any étale morphism  $V \rightarrow X$ , let  $\mathcal{M}_0(V) \subseteq \mathcal{M}(V)$  be the subset consisting of sections  $m \in \mathcal{M}(V)$  such that for any closed point  $x \in |X|$  and any diagram

$$(4.3.2) \quad \begin{array}{ccc} \mathrm{Spec} \, k(x) & \longrightarrow & V \\ & \searrow & \downarrow \\ & & X, \end{array}$$

the stalk element at  $x$  represented by  $m$  is contained in the minimal root of  $\mathcal{M}_x$ .

**Lemma 4.3.3.** *For any closed point  $x \in |X|$ ,  $(\mathcal{M}_0)_x$  is equal to the minimal root of  $\mathcal{M}_x$ .*

*Proof.* By Proposition 2.2.8, there is a nonempty open subcurve  $U \subseteq X$  on which  $\mathcal{M}$  is coherent. Since  $\mathcal{M}$  is also torsion-free, this makes  $\mathcal{M}|_U$  a locally free  $\mathcal{O}_U$ -module of finite rank. By Proposition 3.5.8, the minimal root of  $\mathcal{M}_y$  at any closed point  $y \in |U|$  is  $\mathcal{M}_y$  itself. Thus the condition which defines  $\mathcal{M}_0$  above needs only to be checked at points outside of  $U$ .

Let  $x$  be a closed point of  $|X|$ . We show that the stalk  $(\mathcal{M}_0)_x$  contains the minimal root of  $\mathcal{M}_x$ . Choose any element  $m_x$  from the minimal root of  $\mathcal{M}_x$ . There exists an étale neighborhood

$$(4.3.4) \quad \begin{array}{ccc} \mathrm{Spec} \, k & \longrightarrow & V \\ & \searrow & \downarrow \\ & & X, \end{array}$$

and a section  $m \in \mathcal{M}(V)$  which represents  $m_x$ . Let  $x' \in |V|$  be the image of  $\mathrm{Spec} \, k$  in the diagram above. Consider the restriction

$$(4.3.5) \quad m|_{(V \times_X U) \cup \{x'\}} \in \mathcal{M}((V \times_X U) \cup \{x'\}).$$

By definition, this section is contained in the subsheaf  $\mathcal{M}_0$ . Therefore its stalk  $m_x$  is contained in  $(\mathcal{M}_0)_x$ .

We have shown that the minimal root of  $\mathcal{M}_x$  is contained in  $(\mathcal{M}_0)_x$ . The reverse inclusion is obvious.  $\square$

**Lemma 4.3.6.** *The sheaf  $\mathcal{M}_0$  is coherent.*

*Proof.* As in the previous proof, we may find an open subcurve  $U \subseteq X$  on which  $\mathcal{M}$  is coherent. For any  $y \in |U|$ , the minimal root of  $\mathcal{M}_y$  is  $\mathcal{M}_y$  itself. The quotient sheaf  $\mathcal{M}/\mathcal{M}_0$  is a quasi-coherent skyscraper sheaf supported outside of  $U$ . Since  $\mathcal{M}$  and  $\mathcal{M}/\mathcal{M}_0$  are both quasi-coherent,  $\mathcal{M}_0$  is quasi-coherent.

Since  $\mathcal{M}_{0|U}$  is a finitely-generated  $\mathcal{O}_U$ -module, and each stalk  $(\mathcal{M}_0)_x$  with  $x \in |X \setminus U|$  is a finitely-generated  $\mathcal{O}_{X,x}$ -module,  $\mathcal{M}_0$  is a finitely-generated  $\mathcal{O}_X$ -module. Thus  $\mathcal{M}_0$  is coherent.  $\square$

The other two properties that define a root (see Definition 2.2.1) follow for  $\mathcal{M}_0$  from the corresponding properties for the stalks  $(\mathcal{M}_0)_x$ . Likewise, the fact that  $\mathcal{M}_0$  is contained in every root of  $\mathcal{M}$  follows easily from the same property for the stalks  $(\mathcal{M}_0)_x$ . This completes the proof of the theorem.  $\square$

Suppose that  $\mathcal{M}$  is a torsion-free lfgu  $\mathcal{O}_{F^r, X}$ -module and  $\mathcal{M}_0 \subseteq \mathcal{M}$  is a root for  $\mathcal{M}$ . Then, as in subsection 3.4, we can define a left  $\mathcal{O}_{F^r, X}$ -module structure on the coherent sheaf dual  $\mathcal{M}_0^\vee$ . Let  $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \dots \subseteq \mathcal{M}$  be the filtration of Proposition 2.2.3. If  $\phi: \mathcal{M}_0 \rightarrow \mathcal{O}_X$  is an  $\mathcal{O}_X$ -module homomorphism, then  $F^r(\phi)$  is the composition

$$(4.3.7) \quad \begin{array}{ccccc} \mathcal{M}_0 & \longrightarrow & \mathcal{M}_1 & & \mathcal{O}_X \\ & & \downarrow \cong & & \uparrow \cong \\ & & F_X^{r*} \mathcal{M}_0 & \xrightarrow{F_X^{r*}(\phi)} & F_X^{r*} \mathcal{O}_X. \end{array}$$

Note that if  $\phi$  is nonzero, then  $F^r(\phi)$  is also nonzero. (This is evident because the map  $\mathcal{M}_0 \rightarrow \mathcal{M}_1$  in the above diagram is injective.) So the action of  $F^r$  on  $\mathcal{M}_0^\vee$  is injective.

**Theorem 4.3.8.** *Let  $\mathcal{M}$  be a torsion-free lfgu  $\mathcal{O}_{F^r, X}$ -module, and let  $\mathcal{M}_0 \subseteq \mathcal{M}$  be a root for  $\mathcal{M}$ . Let*

$$(4.3.9) \quad M = \mathcal{H}om_{\mathcal{O}_{F^r, X}}(\mathcal{M}, \mathcal{O}_X).$$

*Then the map  $M \rightarrow \mathcal{M}_0^\vee$  given by restriction fits into an exact sequence*

$$(4.3.10) \quad 0 \longrightarrow M \longrightarrow \mathcal{M}_0^\vee \xrightarrow{1-F^r} \mathcal{M}_0^\vee \longrightarrow 0.$$

*Proof.* It suffices to show that the sequence

$$(4.3.11) \quad 0 \longrightarrow M_x \longrightarrow (\mathcal{M}_0^\vee)_x \xrightarrow{1-F^r} (\mathcal{M}_0^\vee)_x \longrightarrow 0.$$

is exact for every closed point  $x \in |X|$ . Note that  $(\mathcal{M}_0^\vee)_x$  is canonically isomorphic to  $((\mathcal{M}_0)_x)^\vee$ , and, by Proposition 4.2.2,  $M_x$  is canonically isomorphic to

$$(4.3.12) \quad \mathcal{H}om_{\mathcal{O}_{X, x}[F^r]}(\mathcal{M}_x, \mathcal{O}_{X, x}).$$

The exactness of (4.3.11) follows from Theorem 3.4.10.  $\square$

**4.4. Cohomology and the Riemann-Hilbert correspondence.** Let  $Y$  be a smooth *projective*  $k$ -curve. Let  $\mathcal{N}$  be a torsion-free lfgu  $\mathcal{O}_{F^r, Y}$ -module, and let  $\mathcal{N}_0 \subseteq \mathcal{N}$  be a root for  $\mathcal{N}$ . The (injective) Frobenius-linear endomorphism of  $\mathcal{N}_0^\vee$  discussed in the previous subsection induces Frobenius-linear maps

$$(4.4.1) \quad H^i(Y, \mathcal{N}_0^\vee) \rightarrow H^i(Y, \mathcal{N}_0^\vee)$$

for  $i = 0, 1$ . Each cohomology group  $H^i(Y, \mathcal{N}_0^\vee)$  is a finite-dimensional  $k$ -vector space. The map (4.4.1) is injective for  $i = 0$ , and thus makes  $H^i(Y, \mathcal{N}_0^\vee)$  a unit  $k[F^r]$ -module. The left  $k[F^r]$ -module  $H^1(Y, \mathcal{N}_0^\vee)$  is not necessarily unit, but it has a natural decomposition

$$(4.4.2) \quad H^1(Y, \mathcal{N}_0^\vee) \cong V^{ss} \oplus V^{nil},$$

where  $V^{ss}$  is a unit  $k[F^r]$ -module and  $V^{nil}$  is a left  $k[F^r]$ -module with a nilpotent  $F^r$ -action. (See Section 1 of [6] for a discussion of this type of decomposition.)

By Proposition 3.0.12, the  $F^r$ -invariant elements of  $H^0(Y, \mathcal{N}_0^\vee)$  form an  $\mathbb{F}_{p^r}$ -subspace whose dimension is the same as the  $k$ -dimension of  $H^0(Y, \mathcal{N}_0^\vee)$ . Likewise, the  $F^r$ -invariant elements of  $H^1(Y, \mathcal{N}_0^\vee)$  form an  $\mathbb{F}_{p^r}$ -vector space whose dimension is  $\dim_k V^{ss}$ .

The Riemann-Hilbert correspondence implies that these elements are in one-to-one correspondence with co-cycles for the dual of  $\mathcal{N}$ .

**Proposition 4.4.3.** *Let  $Y$  be a smooth projective  $k$ -curve, and let  $N$  be a constructible  $\mathbb{F}_{p^r}$ -étale sheaf on  $Y$  whose sections all have open support. Let*

$$(4.4.4) \quad \mathcal{N} = \mathcal{H}om_{\mathbb{F}_{p^r}}(N, \mathcal{O}_Y),$$

*and let  $\mathcal{N}_0$  be a root for  $\mathcal{N}$ . Then the maps*

$$(4.4.5) \quad H^i(Y, N) \rightarrow H^i(Y, \mathcal{N}_0^\vee)$$

*(given by the double-dual homomorphism  $N \rightarrow \mathcal{N}_0^\vee$ ) map the elements of  $H^i(Y, N)$  bijectively onto the  $F^r$ -invariant elements of  $H^i(Y, \mathcal{N}_0^\vee)$ .*

*Proof.* The module  $\mathcal{N}$  is the dual of  $N$  under the Riemann-Hilbert correspondence. By Theorem 4.3.8, there is an exact sequence

$$(4.4.6) \quad 0 \longrightarrow N \longrightarrow \mathcal{N}_0^\vee \xrightarrow{1-F^r} \mathcal{N}_0^\vee \longrightarrow 0$$

which determines an exact sequence of cohomology groups

$$(4.4.7) \quad 0 \longrightarrow H^0(Y, N) \longrightarrow H^0(Y, \mathcal{N}_0^\vee) \xrightarrow{1-F^r} H^0(Y, \mathcal{N}_0^\vee) \\ \longrightarrow H^1(Y, N) \longrightarrow H^1(Y, \mathcal{N}_0^\vee) \xrightarrow{1-F^r} H^1(Y, \mathcal{N}_0^\vee) \longrightarrow 0.$$

The vector space  $H^0(Y, \mathcal{N}_0^\vee)$  is a trivial left  $k[F^r]$ -module (by Proposition 3.0.12), and it is easily seen that the action of  $(1 - F^r)$  on this module is surjective. So this long exact sequence breaks up into two short exact sequences. The result follows.  $\square$

Proposition 4.4.3 will now enable us to prove the main result of this paper. We establish the following notation: if  $y$  is a closed point of  $Y$ , let  $\mathfrak{C}(N_{(y)})$  denote the minimal root index of the sheaf  $N_{(y)}$  (see Definition 3.5.2). Let  $\chi(Y, N)$  denote the Euler characteristic of  $N$ :

$$(4.4.8) \quad \chi(Y, N) = \dim_{\mathbb{F}_{p^r}} H^0(Y, N) - \dim_{\mathbb{F}_{p^r}} H^1(Y, N).$$

Likewise, if  $\mathcal{Q}$  is a coherent sheaf on  $Y$ , let  $\chi(Y, \mathcal{Q})$  denote the Euler characteristic of  $\mathcal{Q}$ .

**Theorem 4.4.9.** *Let  $Y$  be a smooth projective  $k$ -curve, and let  $N$  be a constructible étale  $\mathbb{F}_{p^r}$ -sheaf on  $Y$  whose sections all have open support. Let  $n$  be the generic rank of  $N$ . Then,*

$$(4.4.10) \quad \chi(Y, N) \geq n \cdot \chi(Y, \mathcal{O}_Y) - \sum_{y \in Y(k)} \mathfrak{C}(N_{(y)}).$$

*Proof.* Let

$$(4.4.11) \quad \mathcal{N} = \mathcal{H}om_{\mathbb{F}_{p^r}}(N, \mathcal{O}_Y),$$

and let  $\mathcal{N}_0 \subseteq \mathcal{N}$  be the unique minimal root for  $\mathcal{N}$ . By Proposition 4.4.3 and the foregoing discussion,

$$(4.4.12) \quad \dim_{\mathbb{F}_{p^r}} H^0(Y, N) = \dim_k H^0(Y, \mathcal{N}_0^\vee)$$

and

$$(4.4.13) \quad \dim_{\mathbb{F}_{p^r}} H^1(Y, N) \leq \dim_k H^1(Y, \mathcal{N}_0^\vee).$$

Therefore,

$$(4.4.14) \quad \chi(Y, N) \geq \chi(Y, \mathcal{N}_0^\vee).$$

Let  $\mathcal{N}_1 \subseteq \mathcal{N}$  be the  $\mathcal{O}_X$ -submodule generated by  $F^r(\mathcal{N}_0)$ . The quotient  $(\mathcal{N}_1/\mathcal{N}_0)$  is a coherent skyscraper sheaf. By definition, the  $k$ -dimension of the stalk of  $(\mathcal{N}_1/\mathcal{N}_0)$  at  $y$  is  $(p^r - 1)\mathfrak{C}(\mathcal{N}_{(y)})$ . Therefore,

$$(4.4.15) \quad \deg \mathcal{N}_1 - \deg \mathcal{N}_0 = (p^r - 1) \sum_{y \in Y(k)} \mathfrak{C}(\mathcal{N}_{(y)}).$$

Meanwhile, the isomorphism  $\mathcal{N}_1 \cong F_X^{r*} \mathcal{N}_0$  implies that  $\deg \mathcal{N}_1 = p^r \deg \mathcal{N}_0$ . Combining these two equalities yields

$$(4.4.16) \quad \deg \mathcal{N}_0 = \sum_{y \in Y(k)} \mathfrak{e}(\mathcal{N}_{(y)}).$$

The desired result now follows using the Riemann-Roch formula:

$$(4.4.17) \quad \chi(Y, N) \geq \chi(Y, \mathcal{N}_0^\vee)$$

$$(4.4.18) \quad = n \cdot \chi(Y, \mathcal{O}_Y) + \deg \mathcal{N}_0^\vee$$

$$(4.4.19) \quad = n \cdot \chi(Y, \mathcal{O}_Y) - \deg \mathcal{N}_0$$

$$(4.4.20) \quad = n \cdot \chi(Y, \mathcal{O}_Y) - \sum_{y \in Y(k)} \mathfrak{e}(N_{(y)}). \quad \square$$

**4.5. Examples.** To illustrate Theorem 4.4.9, we compare three different examples of rank-2 sheaves on the projective line. In the following, we will assume  $p \geq 5$ . If  $Z$  is a  $k$ -curve, let  $\underline{\mathbb{F}}_p$  (or simply  $\underline{\mathbb{F}}_p$ ) denote the constant  $\mathbb{F}_p$ -sheaf on  $Z$ .

Note that if  $f: W \rightarrow W'$  is a finite morphism of smooth projective  $k$ -curves, and  $Q$  is a constructible  $\mathbb{F}_p$ -étale sheaf on  $W$ , then the dimensions of the cohomology groups  $H^i(W', f_* Q)$  are the same as those of  $H^i(W, Q)$ . (This is apparent from a Leray-Serre spectral sequence.)

**Example 4.5.1.** Consider the open immersion  $j: \mathbb{A}_k^1 \setminus \{0\} \rightarrow \mathbb{P}_k^1$ . Let

$$(4.5.2) \quad M = j_* \left( \underline{\mathbb{F}}_p^{\oplus 2} \right).$$

The Euler characteristic of  $M$  is  $-2$ . (This can be proven easily with an exact sequence.) The sheaf

$$(4.5.3) \quad \mathcal{M} = \mathcal{H}om_{\mathbb{F}_p} (M, \mathcal{O}_{\mathbb{P}^1})$$

is isomorphic as a left  $\mathcal{O}_{F^r, \mathbb{P}^1}$ -module to  $j_* \mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}^{\oplus 2}$ . Under this isomorphism, the minimal root  $\mathcal{M}_0 \subseteq \mathcal{M}$  can be identified with the set of sections that have poles of order at most 1 at both 0 and  $\infty$ . If  $\mathcal{M}_1 \subseteq \mathcal{M}$  is the subsheaf generated by  $F^r(\mathcal{M}_0)$ , then

$$(4.5.4) \quad \dim_k (\mathcal{M}_1 / \mathcal{M}_0)_0 = \dim_k (\mathcal{M}_1 / \mathcal{M}_0)_\infty = 2(p-1).$$

The minimal root index for  $M$  at both 0 and  $\infty$  is 2. In this case the formula from Theorem 4.4.9 yields  $\chi(\mathbb{P}^1, M)$  exactly:

$$(4.5.5) \quad 2 \cdot \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) - \mathfrak{e}(M_{(0)}) - \mathfrak{e}(M_{(\infty)}) = 2 \cdot 1 - 2 - 2 = -2.$$

**Example 4.5.6.** Let  $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be a degree-2 morphism which maps 0 to 0 and  $\infty$  to  $\infty$  and is ramified at both of those points. Let  $N = f_* \underline{\mathbb{F}}_p$ . Then  $\chi(\mathbb{P}^1, N) = \chi(\mathbb{P}^1, \mathbb{F}_p) = 1$ .

The sheaf  $N$  is locally constant away from the ramified points of  $f$ . So for any closed point  $x \notin \{0, \infty\}$ , the local sheaf  $N_{(x)}$  is isomorphic to  $\left( \underline{\mathbb{F}}_p \right)^{\oplus 2}$ . The sheaf  $N_{(\infty)}$  is a nontrivial rank-2 sheaf which can be trivialized by a quadratic extension of  $\mathcal{O}_{\mathbb{P}^1, \infty}$ . The reader may verify that there is a simple decomposition

$$(4.5.7) \quad N_{(\infty)} \cong \left( \underline{\mathbb{F}}_p \right) \oplus T,$$



where  $T$  is the nontrivial rank-1 sheaf which arose in Example 3.5.5. By the calculation in that example (and by the fact that the minimal root index is clearly additive over direct sums), we find

$$(4.5.8) \quad \mathfrak{C}(N_{(\infty)}) = \frac{1}{2}.$$

A similar calculation shows that the minimal root index of  $N_{(0)}$  is  $\frac{1}{2}$ . So the formula from Theorem 4.4.9 yields

$$(4.5.9) \quad 2 \cdot \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) - \mathfrak{C}(N_{(0)}) - \mathfrak{C}(N_{(\infty)}) = 2 \cdot 1 - \frac{1}{2} - \frac{1}{2} = 1,$$

which is equal to  $\chi(\mathbb{P}^1, N)$ .

**Example 4.5.10.** Let  $E$  be an elliptic curve, and suppose that  $g: E \rightarrow \mathbb{P}^1$  is a degree-2 morphism which is ramified at 4 distinct points in  $\mathbb{P}^1$ . Let  $P = g_* \left( \underline{\mathbb{F}_p} \right)$ . Let  $a_1, a_2, a_3, a_4 \in \mathbb{P}^1$  be the ramified points of  $g$ . A calculation similar to the one in Example 4.5.6 shows that

$$(4.5.11) \quad \mathfrak{C}(P_{(a_i)}) = \frac{1}{2}.$$

Note that  $\chi(\mathbb{P}^1, P)$  is equal to  $\chi(E, \mathbb{F}_p)$ , which can be 0 or 1, depending on whether  $E$  is supersingular. The lower bound for  $\chi(\mathbb{P}^1, P)$  given by Theorem 4.4.9 is

$$(4.5.12) \quad 2 \cdot \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) - \sum_{i=1}^4 \mathfrak{C}(P_{(a_i)}) = 0.$$

So equality occurs in this case if and only if  $E$  is an ordinary elliptic curve.

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